Conformal Collineations and Ricci Inheritance Symmetry in String Cloud and String Fluids

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Abstract

Conformal collineations (a generalization of conformal motion) and Ricci inheritance collineations, defined by $\pounds_\xi R_{ab} = 2\alpha R_{ab}$, for string cloud and string fluids in general relativity are studied. By investigating the kinematical and dynamical properties of such fluids and using the field equations, some recent studies on the restrictions imposed by conformal collineations are extended, and new results are found.

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1   Introduction

Let M denotes a smooth n-dimensional semi-Riemannian manifold with a non-degenerate metric g of arbitrary signature. Throughout this paper, all geometric objects and their associated structures on M will be assumed smooth.

A space-time on M admits a one-parameter group of conformal motions (Conf M) generated by a vector field ξ if

$$\mathcal{L}_\xi g_{ab} = 2\psi g_{ab}, \tag{1}$$

where $\mathcal{L}_\xi$ signifies the Lie derivative operator along ξ, and $\psi = \psi(x^a)$ is the conformal factor. The vector field ξ is, then, called a conformal Killing vector (CKV) field which is a homothetic vector (HV) if $\psi_{,a} = 0$ and Killing vector (KV) if $\psi = 0$. A special conformal Killing vector (SCKV) field is restricted to $\psi_{;ab} = 0$. Here, $(;)$ and $(,)$ denote the covariant and ordinary derivatives, respectively. A Conf M defined by Eq. (1) satisfies

$$\mathcal{L}_\xi \Gamma^a_{bc} = \delta^a_b \psi_{;c} + \delta^a_c \psi_{;b} - g_{bc} g^{ad} \psi_{;d}, \tag{2}$$

but the converse need not be true. A vector field ξ satisfying (2) is called an affine conformal vector (ACV), which is equivalent to [1, 2]

$$\mathcal{L}_\xi g_{ab} = 2\psi g_{ab} + H_{ab}, \quad H_{[ab]} = H_{ab;c} = 0, \tag{3}$$

where H_{ab} is a symmetric, covariant constant (and, therefore, Killing) tensor associated with ξ. When $\psi_{;ab} = 0$, an ACV is called special affine conformal vector (SACV) field. A vector field ξ satisfying (2) or equivalently (3) is also called *conformal collineation* (Conf C) [3]. A Conf C is reduced to CKV if M is an irreducible Riemannian manifold or a compact orientable Riemannian manifold without boundary or a space of constant curvature. For a Conf C defined by (2), equivalently (3), the following holds [4]

$$\mathcal{L}_\xi g^{ab} = -2\psi g^{ab} - H^{ab}, \tag{4}$$

$$\mathcal{L}_\xi R_{ab} = -(n-2)\psi_{;ab} - g_{ab} \Box \psi, \tag{5}$$

$$\mathcal{L}_\xi R^a_b = -(n-2) g^{ac} \psi_{;cb} - \delta^a_b \Box \psi - 2\psi R^a_b - H^a_c R^c_b, \tag{6}$$

$$\mathcal{L}_\xi R = -2(n-1)\Box \psi - 2\psi R - R', \tag{7}$$

where $\Box$ is the Laplacian operator defined by $\Box \psi = g^{ab} \psi_{;ab}$, and $R' \equiv H^{ab} R_{ab}$.

Recently, Duggal introduced a new symmetry called *Ricci inheritance collineation* (RIC) defined by [5]

$$\mathcal{L}_\xi R_{ab} = 2\alpha R_{ab} \tag{8}$$

where $\alpha = \alpha(x^a)$ is a scalar function. In particular, RIC is reduced to *Ricci collineation* (RC) when $\alpha = 0$. Then, using (3) and (8), we get the following identities

$$\mathcal{L}_\xi R^a_b = 2(\alpha - \psi) R^a_b - H^a_c R^c_b, \tag{9}$$

$$\mathcal{L}_\xi R = 2(\alpha - \psi) R - R'. \tag{10}$$

It follows from (7) and (10) that

$$\Box \psi + \frac{\alpha}{n-1} R = 0. \tag{11}$$

Thus, equating (5) and (8), using this result we have

$$\psi_{;ab} = \frac{\alpha}{n-2} \left[\frac{R}{n-1} g_{ab} - 2R_{ab} \right]. \tag{12}$$

The study of The CKVs and ACVs in a class of fluid space-times has recently attracted some interest. Herrera *et al.*[6] studied CKVs and anisotropic fluids; Duggal and Sharma[1] extended this work to the more general case of special ACV ξ^a. Maartens *et al.*[7] investigated the CKVs in anisotropic fluids, in which they are particularly concerned with SCKVs. Mason and Maartens[8] considered kinematics and dynamics of conformal collineations with the general class of anisotropic fluids and no energy flux. Coley and Tupper[9] discussed space-times admitting SCKV and symmetry inheritance. Yavuz and Yılmaz[10], and Yılmaz *et al.*[11] have examined kinematic and dynamic properties of string cloud and string fluids that admit a Conf M. Also, Baysal *et al.*[12] have been interested in Conf C for string cloud, in the case of a SACVs. In this paper, we will extend the previous obtained results[11, 12] to Conf C for string cloud and string fluids, considering proper ACVs and RICs defined by (8).

The energy-momentum tensor for a cloud of string is

$$T_{ab} = \rho u_a u_b - \lambda \chi_a \chi_b \tag{13}$$

where ρ is the rest energy for cloud of strings with particles attached to them and λ is string tensor density and are related by

$$\rho = \rho_p + \lambda. \tag{14}$$

Here ρ_p is particle energy density. The unit timelike vector u^a describes the cloud four-velocity and the unit spacelike vector χ^a represents a direction of anisotropy, i.e., the string's directions [13]. For u^a and χ^a, we have

$$u^a u_a = -\chi^a \chi_a = -1, \qquad u^a \chi_a = 0. \tag{15}$$

The energy-momentum tensor for a fluid of strings[14, 15] is

$$T_{ab} = (q + \rho_s)(u_a u_b - \chi_a \chi_b) + q g_{ab}, \tag{16}$$

where ρ_s is string density and q is *string tension* and also *pressure*.

The paper is organized as the following. In Sec. 22, some kinematical results are summarized in the case of Conf C. In Sec. 33, using the field equations, dynamical properties and equations of state for string cloud and string fluid are derived by imposing a proper ACV and RIC. In final Section, the obtained results are discussed.

2 Some Kinematical Results

The effect of ACV on any non-null unit vector X^a is given by

$$\pounds_\xi X^a = -\left(\psi + \frac{\epsilon}{2} H_{bc} X^b X^c\right) X^a + Y^a, \tag{17}$$

$$\pounds_\xi X_a = \left(\psi - \frac{\epsilon}{2} H_{bc} X^b X^c\right) X_a + H_{ab} X^b + Y_a, \tag{18}$$

where Y^a is some vector orthogonal to X^a, i.e., $X^a Y_a = 0$, $\epsilon = +1$ if X^a is spacelike and $\epsilon = -1$ if X^a is timelike. The proof of Eqs. (17) and (18) which is a generalization of that for the case of a CKV (see Ref. 7) is obtained by Mason and Maartens[8]. In general $Y^a \neq 0$: an explicit example of a CKV ($H_{ab} = 0$) with $Y^a \neq 0$ in Robertson-Walker spacetime is given in Ref. 7.

For the timelike unit four-velocity vector u^a and spacelike unit vector χ^a of the string cloud and string fluids, we have

$$\pounds_\xi u^a = -\left(\psi - \frac{1}{2} H_{bc} u^b u^c\right) u^a + v^a, \tag{19}$$

$$\pounds_\xi u_a = \left(\psi + \frac{1}{2} H_{bc} u^b u^c \right) u_a + H_{ab} u^b + v_a, \tag{20}$$

where $u^a u_a = -1, \quad u_a v^a = 0$ and

$$\pounds_\xi \chi^a = -\left(\psi + \frac{1}{2} H_{bc} \chi^b \chi^c \right) \chi^a + m^a, \tag{21}$$

$$\pounds_\xi \chi_a = \left(\psi - \frac{1}{2} H_{bc} \chi^b \chi^c \right) \chi_a + H_{ab} \chi^b + m_a, \tag{22}$$

where $\chi^a \chi_a = 1, \quad \chi_a m^a = 0$. Since $\chi_a u^a = 0$ [see eq.(15)] we get

$$\chi_a \pounds_\xi u^a + u^a \pounds_\xi \chi_a = 0. \tag{23}$$

Substituting Eqs. (19) and (22) into (23), yields

$$m_a u^a + v_a \chi^a = -H_{ab} u^a \chi^b. \tag{24}$$

By (19) and (20), it follows that the integral curves of an ACV ξ are material curves iff $v^a = 0$.

3   Dynamical Results and Equations of State

Let (V_4, g_{ab}) be a spacetime of general relativity. Then, we now consider how the Einstein's field equations alter the purely kinematic results of the above section for a Conf C. If ξ^a is an ACV satisfying (3), then

$$\pounds_\xi R_{ab} = -2\psi_{;ab} - g_{ab} \Box \psi, \tag{25}$$

$$\pounds_\xi R = -2\psi R - 6\Box \psi - R', \tag{26}$$

where R_{ab} is Ricci tensor and $R = g^{ab} R_{ab}$ is Ricci scalar. Via Einstein's field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab} \tag{27}$$

we find for the Lie derivative of an arbitrary energy-momentum tensor T_{ab}

$$\pounds_\xi T_{ab} = -2\psi_{;ab} + 2g_{ab} \Box \psi + \frac{1}{2} \left(g_{ab} R' - H_{ab} R \right). \tag{28}$$

By the use of (25) in the known identity[16]

$$\left(R^{ab} \xi_b \right)_{;a} \equiv \frac{1}{2} g^{ab} \pounds_\xi R_{ab} \tag{29}$$

we obtain that

$$\left(R^{ab} \xi_b \right)_{;a} = \frac{1}{2} \left(\pounds_\xi R + 2\psi R + R' \right). \tag{30}$$

If ξ is a ACV satisfying (3), then it follows by (26) in the (30) that

$$\left(R^{ab} \xi_b \right)_{;a} = -3\Box \psi, \tag{31}$$

and if ξ is also RIC, then substituting $\Box \psi = -\frac{\alpha}{3} R$ [see Eq. (11)] in (31), gives

$$\left(R^{ab} \xi_b \right)_{;a} = \alpha R. \tag{32}$$

By means of the Einstein field equations in the form

$$R^{ab} = T^{ab} - \frac{1}{2} T g^{ab} \tag{33}$$

the condition (31) is expressible in the form of conserved 4-current

$$J^a_{;a} = 0, \tag{34}$$

where the conserved vector J^a is defined by

$$J^a = \left(T^{ab} - \frac{1}{2}Tg^{ab}\right)\xi_b + 3g^{ab}\psi_{,b}. \tag{35}$$

Equation (32), or equivalently (34), is the basis for generating new equations of state for various matter under consideration.

3.1 String Cloud

Take into account T_{ab} to be of the form (13), with the aid of (19) for $\mathcal{L}_\xi u_a$ and (21) for $\mathcal{L}_\xi \chi_a$, a direct calculation yields

$$\begin{aligned}
\mathcal{L}_\xi T_{ab} &= [\mathcal{L}_\xi \rho + 2\psi\rho]\, u_a u_b - [\mathcal{L}_\xi \lambda + 2\psi\lambda]\, \chi_a \chi_b \\
&\quad + H_{cd}\left(\rho u^c u^d u_a u_b + \lambda\chi^c\chi^d\chi_a\chi_b\right) + H_{bc}\left(\rho u^c u_a - \lambda\chi_a\chi^c\right) \\
&\quad + H_{ac}\left(\rho u_b u^c - \lambda\chi_b\chi^c\right) + 2\rho u_{(a}v_{b)} - 2\lambda\chi_{(a}m_{b)},
\end{aligned} \tag{36}$$

which, when substituted into (28), gives

$$\begin{aligned}
-2\psi_{;ab} - \frac{1}{2}RH_{ab} + (2\Box\psi + \frac{1}{2}R')g_{ab} &= [\mathcal{L}_\xi \rho + 2\psi\rho]\, u_a u_b - [\mathcal{L}_\xi \lambda + 2\psi\lambda]\, \chi_a \chi_b \\
&\quad + \rho\left[H_{cd}u^d u_a u_b + H_{bc}u_a + H_{ac}u_b\right]u^c \\
&\quad + \lambda\left[H_{cd}\chi^d\chi_a\chi_b - H_{bc}\chi_a - H_{ac}\chi_b\right]\chi^c \\
&\quad + 2\rho u_{(a}v_{b)} - 2\lambda\chi_{(a}m_{b)}.
\end{aligned} \tag{37}$$

Now, we use p_{ab} projection tensor that projects in the directions that are perpendicular to both χ^a and u^a,

$$p_{ab} = g_{ab} + u_a u_b - \chi_a \chi_b. \tag{38}$$

Some properties of this tensor are

$$p^{ab}u_b = p^{ab}\chi_b = 0 \tag{39}$$

$$p^a_c p^c_b = p^a_b, \;\; p_{ab} = p_{ba} \tag{40}$$

By contracting (37) with the tensors $u^a u^b, \chi^a\chi^b, p^{ab}, u^a\chi^b, u^a p^{bc}, \chi^a p^{bc}$, and $p^a_c p^b_d - \frac{1}{2}p^{ab}p_{cd}$ the following equations for the string cloud are derived:

$$\mathcal{L}_\xi \rho + 2\psi\rho = -2\left(\Box\psi + \psi_{;ab}u^a u^b\right) \;\; + \;\; \frac{1}{2}\rho_p H_{ab}u^a u^b - \frac{1}{2}R' \tag{41}$$

$$\mathcal{L}_\xi \lambda + 2\psi\lambda = -2\left(\Box\psi - \psi_{;ab}\chi^a\chi^b\right) \;\; + \;\; \frac{1}{2}\rho_p H_{ab}\chi^a\chi^b - \frac{1}{2}R' \tag{42}$$

$$\frac{1}{2}\left[(\rho+\lambda)H_{ab} + 4\psi_{;ab}\right]p^{ab} \;\; = \;\; 4\Box\psi + R', \tag{43}$$

$$\rho_p\left[\chi^a v_a + \frac{1}{2}H_{ab}u^a\chi^b\right] \;\; = \;\; 2\psi_{;ab}u^a\chi^b, \tag{44}$$

$$\rho p^{bc}v_b + \frac{1}{2}\rho_p H_{ab}u^a p^{bc} \;\; = \;\; 2\psi_{;ab}u^a p^{bc}, \tag{45}$$

$$\lambda p^{bc}m_b - \frac{1}{2}\rho_p H_{ab}\chi^a p^{bc} \;\; = \;\; 2\psi_{;ab}\chi^a p^{bc}, \tag{46}$$

$$[(\rho+\lambda)H_{ab} + 4\psi_{;ab}]\left(p^a_c p^b_d - \frac{1}{2}p^{ab}p_{cd}\right) \;\; = \;\; 0, \tag{47}$$

where $R' = R^{cd}H_{cd}$, that is,

$$R' = \frac{1}{2}\left[(\rho + \lambda)p^{cd} + \rho_p(u^c u^d + \chi^c \chi^d)\right] H_{cd}. \tag{48}$$

Equations (41)-(47) are valid for any ACV ξ^a. Using (43) in (47) we have

$$\left[(\rho + \lambda)H_{ab} + 4\psi_{;ab}\right] p_c^a p_d^b = (4\Box\psi + R')p_{cd}, \tag{49}$$

which gives Eq. (43) by contracting with g^{cd}. Therefore, Eqs. (43) and (47) are equivalent.

Now, using the string cloud energy-momentum tensor (13), it follows from Einstein's field equations (27) that

$$R_{ab}u^a u^b = \frac{1}{2}\rho_p = R_{ab}\chi^a \chi^b, \quad R_{ab}p^{ab} = \rho + \lambda. \tag{50}$$

Then, Eqs. (11), (12), and (50) give

$$\psi_{;ab}u^a u^b = \frac{\alpha}{3}(\lambda - 2\rho), \ \psi_{;ab}\chi^a \chi^b = \frac{\alpha}{3}(2\lambda - \rho), \ \psi_{;ab}p^{ab} = -\frac{2\alpha}{3}(\rho + \lambda), \tag{51}$$

$$\psi_{;ab}u^a \chi^b = 0, \ \psi_{;ab}u^a p^{bc} = 0, \ \psi_{;ab}\chi^a p^{bc} = 0. \tag{52}$$

Thus, substituting $\Box\psi = -\frac{\alpha}{3}(\rho + \lambda)$ [see Eq. (11)] and (51) into Eqs. (41), (42) and (43), yield

$$\pounds_\xi \rho + 2(\psi - \alpha)\rho = -\frac{1}{4}H_{ab}\left[(\rho + \lambda)p^{ab} - 2\rho_p u^a u^b\right], \tag{53}$$

$$\pounds_\xi \lambda + 2(\psi - \alpha)\lambda = -\frac{1}{4}H_{ab}\left[(\rho + \lambda)p^{ab} - 2\rho_p \chi^a \chi^b\right], \tag{54}$$

$$R' = \frac{1}{2}(\rho + \lambda)H_{ab}p^{ab}. \tag{55}$$

Using (55) in (48), we get

$$\rho_p\left(u^a u^b + \chi^a \chi^b\right)H_{ab} = 0. \tag{56}$$

Furthermore, substituting (52) into equations (44)-(46), it follows that

$$\rho_p\left[\chi^a v_a + \frac{1}{2}H_{ab}u^a \chi^b\right] = 0, \tag{57}$$

$$\rho p^{bc}v_b + \frac{1}{2}\rho_p H_{ab}u^a p^{bc} = 0, \tag{58}$$

$$\lambda p^{bc}m_b - \frac{1}{2}\rho_p H_{ab}\chi^a p^{bc} = 0. \tag{59}$$

For a cloud of strings with T_{ab} given in (13), using $R = (\rho + \lambda)$, we find from (32) that

$$2\alpha(\rho + \lambda) = \left(\left[(\rho + \alpha)p^{ab} + \rho_p(u^a u^b + \chi^a \chi^b)\right]\xi_b\right)_{;a}. \tag{60}$$

Then, following three cases are taken (where we use the relation $\xi^a_{;a} = 4\psi + \frac{1}{2}H^a_a$ obtained from (3)) :

Case (i) : If ξ^a is orthogonal to u^a, i.e. $u^a \xi_a = 0$, thus, Eq. (60) provides

$$2\alpha(\rho + \lambda) - 4\psi\rho_p = \pounds_\xi \rho - \pounds_\xi \lambda + \frac{1}{2}\rho_p H^a_a \tag{61}$$

Then, substituting (53) and (54) into this equation, after some algebra, yields

$$2\alpha\lambda = \rho_p\left[\psi + \frac{1}{4}H^a_a + \frac{1}{4}H_{ab}(u^a u^b - \chi^a \chi^b)\right]$$

which, using (56), becomes

$$8\alpha\lambda = \rho_p\left[4\psi + H^a_a - 2H_{ab}\chi^a \chi^b\right]. \tag{62}$$

Case (ii) : If ξ^a is parallel to u^a, i.e. $\xi^a = \xi u^a$, then since $u^a \chi_a = 0$ and $p^{ab} u_a = 0$, we obtain from equation (60) that

$$2\alpha(\rho + \lambda) + 4\psi \rho_p = \pounds_\xi \lambda - \pounds_\xi \rho - \frac{1}{2} \rho_p H_a^a \tag{63}$$

which, when (53) and (54) substituted into (63), and using (56), gives

$$8\alpha\rho = -\rho_p \left[4\psi + H_a^a + 2H_{ab} u^a u^b \right]. \tag{64}$$

Case (iii) : If ξ^a is orthogonal to both u^a and χ^a, then $p^{ab}\xi_b = \xi^a$, and Eq. (60) becomes

$$2(\alpha - 2\psi)(\rho + \lambda) = \pounds_\xi \rho + \pounds_\xi \lambda + \frac{1}{2}(\rho + \lambda) H_a^a \tag{65}$$

Proceeding as before we find

$$(\rho + \lambda) \left[4\psi + H_a^a - H_{ab} p^{ab} \right] = 0. \tag{66}$$

Let us consider the following possibilities according to whether ρ_p vanishes or not.

Case (A) : $\rho_p = 0$, i.e. $\rho = \lambda$ (geometric string).

In this case, for $\rho \neq 0 \neq \lambda$, we have from Eqs. (53)-(60) that

$$\pounds_\xi \rho + 2(\psi - \alpha) = -\frac{1}{2} H_{ab} p^{ab} \rho, \tag{67}$$

$$v^a = (\chi^b v_b) \chi^a, \tag{68}$$

$$m^a = -(u^b m_b) u^a, \tag{69}$$

$$2\alpha\rho = \left(\rho p^{ab} \xi_b \right)_{;a}. \tag{70}$$

When $\xi^a \perp u^a = 0$, i.e. $\xi^a u_a$, or $\xi^a \parallel u^a$, i.e., $\xi^a = \xi u^a$, we find from the above equations that $\alpha = 0$ and $\psi_{;ab} = 0$ so that RIC is reduced to RC, that is, string cloud does not admit RIC, and ξ^a must be a SACV field. If ξ^a is orthogonal to both u^a and χ^a, then from (67) and (70), we obtain

$$\pounds_\xi \rho + 2 \left(\frac{1}{2} H_{ab} p^{ab} - \frac{1}{4} H_a^a - \alpha \right) \rho = 0, \tag{71}$$

$$\psi = \frac{1}{4} \left(H_{ab} p^{ab} - H_a^a \right), \tag{72}$$

which, assuming $H_{ab} u^b = \mu u_a$, i.e. u^a is an eigenvector of H_{ab} and $H_{ab} \chi^b = \mu \chi_a$, i.e. χ^a is an eigenvector of H_{ab}, yield

$$\pounds_\xi \rho + 2 \left(\frac{1}{4} H_a^a - \mu - \alpha \right) \rho = 0, \tag{73}$$

$$\mu = -2\psi. \tag{74}$$

where μ is an eigenvalue of H_{ab}.

Case (B) : $\rho_p \neq 0$.

In this case, it is assumed that

$$\rho \neq 0 \neq \lambda, \tag{75}$$

and

$$H_{ab} u^b = \mu u_a, \tag{76}$$

then from Eqs. (24) and (56)-(59), we have

$$H_{ab} \chi^b = \mu \chi_a \tag{77}$$

and

$$v^a = 0 = m^a. \tag{78}$$

In this case, there are also three subcases :

Subcase (B.i) : $\xi^a \perp u^a$, i.e. $\xi^a u_a = 0$. In this subcase, using the above obtained results in (53), (54), and (62), we find

$$\pounds_\xi \rho + 2\left(\psi - \alpha - \frac{1}{8}H_a^a\right)\rho = \left(\mu - \frac{1}{4}H_a^a\right)\lambda, \tag{79}$$

$$\pounds_\xi \lambda + 2\left(\psi - \alpha - \frac{1}{8}H_a^a\right)\lambda = \left(\mu - \frac{1}{4}H_a^a\right)\rho, , \tag{80}$$

$$\alpha = \frac{\rho_p}{8\lambda}\left[4\psi - 2\mu + H_a^a\right]. \tag{81}$$

Subcase (B.ii) :$\xi^a \parallel u^a$, i.e., $\xi^a = \xi u^a$. In this subcase, it is seen that $\pounds_\xi \rho$ and $\pounds_\xi \lambda$ are same as in subcase (B.i). However, from (64), α is

$$\alpha = -\frac{\rho_p}{8\rho}\left[4\psi - 2\mu + H_a^a\right]. \tag{82}$$

Subcase (B.iii) : $\xi^a \perp u^a$ and $\xi^a \perp \chi^a$. For this subcase, it follows from (53), (54), and (66) that

$$\pounds_\xi \rho + 2\left(\frac{1}{8}H_a^a - \frac{1}{2}\mu - \alpha\right)\rho = \left(\mu - \frac{1}{4}H_a^a\right)\lambda, \tag{83}$$

$$\pounds_\xi \lambda + 2\left(\frac{1}{8}H_a^a - \frac{1}{2}\mu - \alpha\right)\lambda = \left(\mu - \frac{1}{4}H_a^a\right)\rho, , \tag{84}$$

$$\mu = -2\psi. \tag{85}$$

3.2   String Fluid

In this subsection, following exactly as in the case of string cloud, taking the Lie derivative of T_{ab} given by (16) for string fluid and contracting in turn by the tensors $u^a u^b$, $\chi^a \chi^b$, p^{ab}, $u^a \chi^b$, $u^a p^{bc}$,$\chi^a p^{bc}$,and $p_c^a p_d^b - \frac{1}{2}p^{ab}p_{cd}$, after a number of calculations, we get

$$\pounds_\xi \rho_s + 2\psi\rho_s = -2\square\psi - [2\psi_{;ab} - qH_{ab}]u^a u^b - \frac{1}{2}R' \tag{86}$$

$$\pounds_\xi \rho_s + 2\psi\rho_s = -2\square\psi + [2\psi_{;ab} - qH_{ab}]\chi^a \chi^b - \frac{1}{2}R' \tag{87}$$

$$\pounds_\xi q + 2\psi q = 2\square\psi - \frac{1}{2}[2\psi_{;ab} + \rho_s H_{ab}]p^{ab} + \frac{1}{2}R', \tag{88}$$

$$[2\psi_{;ab} - qH_{ab}]u^a \chi^b = 0, \tag{89}$$

$$(\rho_s + q)p^{bc}v_b = [2\psi_{;ab} - qH_{ab}]u^a p^{bc}, \tag{90}$$

$$(\rho_s + q)p^{bc}m_b = [2\psi_{;ab} - qH_{ab}]\chi^a p^{bc}, \tag{91}$$

$$[2\psi_{;ab} + \rho_s H_{ab}]\left(p_c^a p_d^b - \frac{1}{2}p^{ab}p_{cd}\right) = 0, \tag{92}$$

where $R' = R^{cd}H_{cd}$, that is,

$$R' = (\rho_s + q)H_{cd}p^{cd} - qH_a^a. \tag{93}$$

Equating Eqs. (86) and (87), we find

$$[2\psi_{;ab} - qH_{ab}]\left(u^a u^b + \chi^a \chi^b\right) = 0. \tag{94}$$

Using (16) in (27), the following relations are derived

$$R_{ab}u^a u^b = q = -R_{ab}\chi^a \chi^b, \quad R_{ab}p^{ab} = 2(\rho_s - q) \tag{95}$$

which, from Eqs. (11) and (12), yields

$$\psi_{;ab}u^a u^b = -\frac{\alpha}{3}(\rho_s + 2q) = -\psi_{;ab}\chi^a \chi^b, \quad \psi_{;ab}p^{ab} = -\frac{2\alpha}{3}(2\rho_s + q), \tag{96}$$

$$\psi_{;ab}u^a \chi^b = 0, \quad \psi_{;ab}u^a p^{bc} = 0, \quad \psi_{;ab}\chi^a p^{bc} = 0. \tag{97}$$

Then, substituting $\Box\psi = -\frac{2\alpha}{3}(\rho_s - q)$ [see Eq. (11)], (96) and (97) into Eqs. (86)-(94), gives

$$\pounds_\xi \rho_s + 2(\psi - \alpha)\rho_s = -\frac{1}{2}\rho_s H_{ab}p^{ab}, \tag{98}$$

$$\pounds_\xi q + 2(\psi - \alpha)q = \frac{1}{2}q\left[H_{ab}p^{ab} - H_a^a\right], \tag{99}$$

$$qH_{ab}\left(u^a u^b + \chi^a \chi^b\right) = 0, \tag{100}$$

$$qH_{ab}u^a \chi^b = 0, \tag{101}$$

$$(\rho_s + q)p^{bc}v_b = -qH_{ab}u^a p^{bc}, \tag{102}$$

$$(\rho_s + q)p^{bc}m_b = -qH_{ab}\chi^a p^{bc}, \tag{103}$$

$$\rho_s H_{ab}\left(p_c^a p_d^b - \frac{1}{2}p^{ab}p_{cd}\right) = 0. \tag{104}$$

For a fluid of strings with T_{ab} given in (16), using $R = 2(\rho_s - q)$ in (32), we get

$$2\alpha(\rho_s - q) = \left(\left[(\rho_s + q)p^{ab} - qg^{ab}\right]\xi_b\right)_{;a}. \tag{105}$$

Now, we consider the cases (i)-(iii) given in string cloud. In both cases (i), $\xi^a u_a = 0$, and (ii), $\xi^a = \xi u^a$, Eqs. (98), (99), and (105) yield

$$4(\alpha\rho_s + \psi q) = -qH_{ab}p^{ab}. \tag{106}$$

In case (iii), $\xi^a u_a = 0 = \xi^a \chi_a$, from (98), (99), and (105), we find

$$4(\alpha q + \psi\rho_s) = \rho_s H_{ab}\left(u^a u^b - \chi^a \chi^b\right). \tag{107}$$

Thus, in the case of geometric string, that is,

$$\rho_p = 0, \quad and \quad q = 0, \quad i.e. \ \rho = \rho_s = \lambda \tag{108}$$

we obtain the same relations given in (67)-(70). If we assume

$$q \neq 0 \neq \rho_s \tag{109}$$

and

$$H_{ab}u^b = \mu\, u_a, \tag{110}$$

then, it is found from (24), (98)-(104) that

$$\pounds_\xi \rho_s + 2\left(\psi - \alpha - \frac{1}{2}\mu + \frac{1}{4}H_a^a\right)\rho_s = 0, \tag{111}$$

$$\pounds_\xi q + 2\left(\psi - \alpha + \frac{1}{2}\mu\right)q = 0, \tag{112}$$

$$H_{ab}\chi^b = \mu\chi^a, \quad u^a m_a = -\chi^a v_a, \tag{113}$$

$$v^a = (\chi^a v_b)\chi^a, \quad m^a = -(u^b m_b)u^a. \tag{114}$$

Therefore, using these results, we obtain from Eq. (106) that

$$\alpha = \frac{q}{4\rho_s}\left[4\psi - 2\mu + H_a^a\right], \tag{115}$$

for cases (i) and (ii), and from Eq. (107)

$$\alpha = -\left(\psi + \frac{1}{2}\mu\right)\frac{\rho_s}{q}. \tag{116}$$

for case (iii). The derivations of (111)-(116) required the reasonable assumptions contained in (109) since, when $\rho_s + q = 0$, fluid of strings disappear.

4 Discussions and Conclusions

In this paper, we have used the Einstein field equations to investigate the dynamic restrictions imposed by the proper ACV and RIC, and derived equations of state for string cloud and string fluids. In order to find physically meaningful equations of state, we need a relation between α, ψ and H_a^a. For this purpose, we define *symmetry inheritance* of an ACV to mean that a kinematical or a dynamical variable F, say, satisfies an equation of the form

$$\pounds_\xi F + (k\psi + r\alpha + sH_a^a)F = 0, \tag{117}$$

where k, r and s are constants, possibly depending on the tensorial character of F. Symmetry inheritance for a CKV in the above sense has been considered in Refs. 9 and 17. It is an appealing idea because it relates the symmetry with all kinematical and dynamical variables making full use of the field equations. If F satisfies Eq. (117), then the symmetries of an ACV ξ^a are inherited. If this condition does not hold, then the symmetries are not inherited. Also, in both of the cases of string cloud and string fluids, we conclude that in order for the dynamical variables to inherit the symmetry of an ACV, the value of the constant r is -2. The constants k and s behave differently for considered cases.

(a) In the case of string cloud, we have found Eqs. (62), (64) and (66) from the restrictions (i), (ii), and (iii) on the equation of state, imposed by the field equations. Then, in case (A), i.e. $\rho = \lambda$ (geometric string), we have also found $\alpha = 0$ and $\psi_{;ab} = 0$ when $\xi^a \perp u^a$ and $\xi^a = \xi u^a$, so that there exist no RIC other then RC, that is, string cloud does not admit RIC. When $\xi^a u_a = 0 = \xi^a \chi_a$, assuming u^a and χ^a are eigenvectors of H_{ab}, it is obtained (71) as inheritance equation. In this case, if we assume that

$$\mu = \frac{1}{4}H_a^a, \tag{118}$$

then our symmetry inheritance equation (117) are reduced to the same definitions in Refs. 9 and 17. Also, in case (A), it is found that v^a and m^a are spacelike and timelike vector fields, respectively. In case (B), i.e. $\rho_p \neq 0$, it is obtained that there are two important cases when u^a and χ^a are eigenvectors of H_{ab}, so that v^a and m^a vanish by (78) (provided the physically reasonable energy condition (75) is satisfied). When $\alpha \neq 0$ (i.e., proper RIC), Eqs. (62), (64) and (66) provide physically meaningful equations of state for given α, ψ, and H_a^a.

In subcase (B.i), under the assumption (118), from Eqs. (79), (80) and (81), we obtain that

$$\pounds_\xi \rho + 2\left(\psi - \alpha - \frac{1}{8}H_a^a\right)\rho = 0, \tag{119}$$

$$\pounds_\xi \lambda + 2\left(\psi - \alpha - \frac{1}{8}H_a^a\right)\lambda = 0, \tag{120}$$

$$\alpha = \frac{\rho_p}{2\lambda}\left[\psi + \frac{1}{8}H_a^a\right]. \tag{121}$$

The equation (121) yields the following equation of state for $\psi \neq -\frac{1}{4}H^a_a$

$$\rho = (1 + w)\lambda \tag{122}$$

which coincide with Takabayashi string, where w is given by

$$w = \frac{2\alpha}{\psi + \frac{1}{8}H^a_a}. \tag{123}$$

In subcase (B.ii), $\mathcal{L}_\xi \rho$ and $\mathcal{L}_\xi \lambda$ are same as in subcase (B.i), but assuming the condition (118), α is

$$\alpha = -\frac{\rho_p}{2\rho}\left[\psi + \frac{1}{8}H^a_a\right] \tag{124}$$

giving equation of state (122) with

$$w = -\frac{2\alpha}{2\alpha + \psi + \frac{1}{8}H^a_a}. \tag{125}$$

In subcase (B.iii), using the condition (118) in (83) and (84), we have

$$\mathcal{L}_\xi \rho - 2\alpha\rho = 0, \tag{126}$$

$$\mathcal{L}_\xi \lambda - 2\alpha\lambda = 0. \tag{127}$$

(b) In the case of string fluids, when $\rho_p = 0$ and $q = 0$, i.e. geometric (Nambu) string, same relations in string cloud are found. When $q \neq 0 \neq \rho_s$, it is found that v^a and m^a are spacelike and timelike vector fields, respectively, and Eqs. (111) and (112) are inheritance equations with $k = 2$, $r = -2$ and $s = \frac{1}{8}$ if the condition (118) holds. Then, assuming the condition (118), it follows from (111) and (112) that

$$\mathcal{L}_\xi \rho_s + 2\left(\psi - \alpha + \frac{1}{8}H^a_a\right)\rho_s = 0, \tag{128}$$

$$\mathcal{L}_\xi q + 2\left(\psi - \alpha + \frac{1}{8}H^a_a\right)q = 0. \tag{129}$$

Hence, in both cases (i) and (ii), we have found the following equation of state

$$q = \gamma\rho_s \tag{130}$$

where we have defined γ as

$$\gamma = \frac{\alpha}{\psi + \frac{1}{4}H^a_a}. \tag{131}$$

with $\psi \neq -\frac{1}{4}H^a_a$. Then, using (131), Eqs. (128) and (129) become

$$\mathcal{L}_\xi \rho_s + 2\left(\frac{1 - \gamma}{\gamma}\right)\alpha\rho_s = 0, \tag{132}$$

$$\mathcal{L}_\xi q + 2\left(\frac{1 - \gamma}{\gamma}\right)\alpha q = 0, \tag{133}$$

where $\gamma \neq 0$. For case (iii), γ is defined as

$$\gamma = -\frac{\psi + \frac{1}{4}H^a_a}{\alpha} \tag{134}$$

giving (130). Therefore, substituting (134) into Eqs. (128) and (129), yields

$$\mathcal{L}_\xi \rho_s - 2\left(1 + \gamma\right)\alpha\rho_s = 0, \tag{135}$$

$$\mathcal{L}_\xi q - 2\left(1 + \gamma\right)\alpha q = 0. \tag{136}$$

Finally, it is easily seen that if $\gamma = 1$, i.e. $q = \rho_s$, in cases (i) and (ii), or if $\gamma = -1$, i.e. $q + \rho_s = 0$, in case (iii), then $\mathcal{L}_\xi \rho_s = 0$ and $\mathcal{L}_\xi q = 0$.

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